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# Cusp Anomalous dimension and rotating open strings in AdS/CFT

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**ABSTRACT:** In the context of AdS/CFT we provide analytical support for the proposed duality between a Wilson loop with a cusp, the cusp anomalous dimension, and the meson model constructed from a rotating open string with high angular momentum. This duality was previously studied using numerical tools in [1]. Our result implies that the minimum of the profile function of the minimal area surface dual to the Wilson loop, is related to the inverse of the bulk penetration of the dual string that hangs from the quark–anti-quark pair (meson) in the gauge theory.

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# 1 Introduction and result

By a clever combination of modern methods of scattering amplitudes [2], the old Regge scattering amplitudes approach (Regge trajectory) [3] and taking advantage of the dual conformal momentum space symmetry of the planar  $\mathcal{N} = 4$  SYM theory in four dimensions, the authors of [1] (see also [4]) obtained a duality between the higher angular momentum state for a given energy  $E$  in terms of the cusp anomalous dimension  $\Gamma_{cusp}$  and the angle  $\theta$  of the cusp of a Wilson line with a cusp given by [4, 5],

$$J + 1 = -\Gamma_{cusp}(\theta), \quad E^2 = 4m^2 \cos^2 \theta/2. \quad (1.1)$$

where  $J$  is the angular momentum,  $m$  is the mass and  $E$  the energy. This duality was scrutinized in the perturbative regime as well as in the strong coupling limit using AdS/CFT, reproducing results of the meson spectrum [6] using numerical methods. Unfortunately, the authors in [1] were unable to find an analytic relation between the string model of the meson, constructed from an open rotating string as the dual of the meson (massive quark anti-quark pair) and the cusp anomalous dimension using the Wilson loop approach. The aim of this paper is to fill this gap and give an explicit analytic expression that relates the parameters of these two very different approaches for the problem of finding the meson spectrum. We will work in the context of AdS/CFT so the window that we will explore is the strong coupling regime in the field theory side.

The central result of our paper is that we can calculate the energy and the angular momentum of the meson model in *closed* form in terms of only one parameter  $\tilde{z}_0$  that correspond to the maximum penetration of the dual string profile in AdS. The result is

$$E(\tilde{z}_0)/m_q = 2 \cos(\theta(\tilde{z}_0)/2), \quad (1.2)$$

$$J(\tilde{z}_0) + 1 = \frac{\sqrt{\lambda}}{4\tilde{z}_0\sqrt{1+\tilde{z}_0^2}} {}_2F_1\left(\frac{1}{2}, \frac{3}{2}, 2, -\frac{1}{\sqrt{1+\tilde{z}_0^2}}\right), \quad (1.3)$$

where

$$\theta(\tilde{z}_0) = 2K \int_0^\infty \frac{d\xi}{(1+\xi^2)\sqrt{(1+\tilde{z}_0^2+\xi^2)(2+\tilde{z}_0^2+\xi^2)}}, \quad (1.4)$$

is the cusp angle,  $E$  is the energy of the quark, anti-quark pair,  $m_q$  the quark mass,  $J$  the angular momentum and  $K = \tilde{z}_0\sqrt{1+\tilde{z}_0^2}$ .  ${}_2F_1$  is the hypergeometric function.

Using these formulas we can reconstruct the information encoded in the meson spectrum. In particular, the energy and angular momentum in parametric form with parameter  $\tilde{z}_0$ . It is usual to represent this result in a parametric  $E - J$  plot. The

plot was reported in [6] and reproduced in [1]. In the context of AdS/CFT duality we provide here a version of this plot (see Fig. 3). In this figure we show that in fact the numerical integration of the meson model is equivalent to our results. Fig. 3 shows the Regge trajectories for different values of  $\lambda$ , the 't Hooft coupling, obtained by numerical integration of the equations of motion for the string profile as compared with the analytic result given by our relations (1.2) and (1.3) in terms of the parameter  $\tilde{z}_0$  that measure the maximum penetration of the string profile in the bulk. Our results reproduce the well know asymptotic analytical information of previous analysis as well as predict new information for the next-to-next to leading order for  $E$  and  $J$  and in fact for the complete functions  $E$  and  $J$ . The information provided by our formulas is equivalent to solve a differential equation for the profile of the string that as far as we know was not solved in analytic form yet. Nevertheless, using the Wilson loop with a cusp angle  $\theta$  and then relating  $\theta$  with the energy and the angular momentum with  $\Gamma_{cusp}$  the cusp anomalous dimension, through the duality (1.1) we can recover all the information encoded in the string profile and the definition of the energy and angular momentum from the bulk of the meson model. Recently another very interesting observable, namely the length of the string in the meson model, was shown to be related to the anomalous dimension of twist two operators in the field theory side [7].

We will review the construction of the central concepts relevant to our work and then present support for our results (1.2) and (1.3). First we review the construction of the meson model from a rotating string in AdS and extract analytic information through the analysis of interesting asymptotic limits. These analytical results are crucial to develop our relation between a Wilson loop with a cusp and meson model. We will present them in section 2 in a form that is well suited for our general proposes. A crucial point relies in the identification of the parameter of the string profile to write the energy and the angular momentum as functions of this string parameter. The identification of this parameter will be given by observing that the corresponding analytical expressions that comes from the duality (1.1), through  $\Gamma_{cusp}$  and the cusp angle  $\theta$  associated with the angular momentum and the energy respectively, match the corresponding expressions of the meson model through the duality (1.1). To that end we will review also the construction of the basic ideas to obtain the cusp anomalous dimension in the context of AdS/CFT (see Sect. 3). In section 4 we present a compact presentation of the dual conformal symmetry of planar  $\mathcal{N} = 4$  SYM amplitudes and their use to relate the cusp anomalous dimension and the cusp angle to the angular momentum and the energy respectively. We refer the reader to the original references for details. In sections 5 and 6, we present our results and a dictionary that gives analytical support to the duality (1.1) in the context of AdS/CFT. Finally in section 7, we present some final comments and provide some ideas for future work.

## 2 Meson spectrum from a rotating string

The AdS/CFT duality is nowadays a basic tool to study some aspects of strongly coupled gauge theories by using a semiclassical gravity solution in asymptotically *AdS* space-time [8]. The best studied case is the duality between  $\mathcal{N} = 4$  SYM theory in four dimensions at strong coupling and a weakly coupled gravitational theory in five dimensional *AdS*. It is well known that to include fields in the *fundamental* representation of  $SU(N)$  (quarks) in AdS/CFT, one must introduce probe D7-branes in *AdS* [9]. The dual gauge theory is the  $\mathcal{N} = 2$  SYM theory with massive quarks. The beta function of this theory is  $\beta \propto \lambda^2 N_f / N$ , where  $N_f$  is the number of flavour D7 branes and  $N$  is the number colour D3 branes. This beta function tends to zero for  $N_f$  small, fixed 't Hooft coupling  $\lambda$  and  $N \rightarrow \infty$  (planar limit), such that the theory remains conformal in this limit.

Mesons of low spin are described as fluctuations of the D7-branes. High spin mesons are represented by semiclassical rotating open strings attached to the D7-branes [6] (see also [10] for a review).

An exact result would require a non-trivial background quantization of open strings attached to the D7-brane. Nevertheless, for large spin operators, the associated anomalous dimension is small and therefore quantum corrections are negligible [11]. The spectrum can be obtained from classical rotating open strings whose end points are attached to the D7-brane. Two opposite regimes appear naturally once one compares the proper size  $\delta$  of the string to the AdS radius  $R \sim (g_s N \alpha'^2)^{1/4}$ . If  $\delta \ll R$  then the curvature of AdS is irrelevant and hence the spectrum should be that found in flat space, i.e., we should find Regge behaviour  $E \sim \sqrt{J}$ . Since in flat space  $\delta \sim \sqrt{J \alpha'}$ , this regime should occur when  $J \ll \sqrt{g_s N}$ . If instead,  $J \gg \sqrt{g_s N}$ , then the AdS curvature becomes important and we expect the spectrum will be drastically modified. Indeed, it agrees with two non-relativistic quarks weakly bound by a Coulomb potential [6].

The presence of the D7-branes introduces a mass scale in the duality, the quark mass  $m_q$ , corresponding to the position of the branes along the radial direction of AdS5. The mass  $m_q$  of the quark in the gauge theory is related to the separation distance  $L = R^2/z_{D7}$  between D3-branes and D7-branes by the relation

$$m_q = \frac{R^2}{2\pi\alpha' z_{D7}} \quad (2.1)$$

where  $z_{D7}$  is the position of the D7 branes in the bulk. The introduction of a mass scale breaks the invariance under dilatations. The isometries of the AdS space-time rescale the radial position of the D7-brane, which in turn corresponds to a redefinition of the quark mass  $m_q$  in the gauge theory. Nevertheless we have a

new isometry (see below) that implies that the observables are appropriate rescaled variables. For example the energy is not an invariant but  $E/m_q$  indeed is an invariant. The separation between the string endpoints on the D7-brane is not an invariant but we can find a rescaled variable that plays the role of the meson size in the gauge theory.

The system we want to describe consists of open strings rotating in  $AdS_5$  with endpoints attached to a probe D7 brane, which are dual to mesons in the  $\mathcal{N} = 2$  Yang Mills theory having a very large space-time spin  $J$ .

In order to describe our string configuration, we work with  $AdS_5$  space using Poincaré coordinates and choose the metric as

$$ds^2 = \frac{R^2}{z^2} (-dt^2 + d\rho^2 + \rho^2 d\theta^2 + dz^2) , \quad (2.2)$$

where the spatial coordinates on the Minkowski boundary are in cylindrical form  $(\rho, \theta)$ . A high-spin meson is represented by a semiclassical rotating string with a steady profile  $\rho = \rho(z), \theta = \omega t$ , which is symmetric around  $\rho = 0$  and reaches a maximum value at  $z = z_0$  in the radial direction. The string endpoints are attached to a D7-brane located at  $z = z_{D7}$ . As remarked above this parameter is a measure of the quark mass  $m_q$ . A very massive meson implies that the D7-brane is near the boundary.

As discussed in [6], it is convenient to use the dimensionless coordinates  $\tilde{\rho} \equiv \omega\rho$ ,  $\tilde{z} \equiv \omega z$ . In the problem of the rotating open string, neither  $\omega$  nor  $z_0$  are AdS scalars, rather they separately change under dilatations, but in such a way that their product remains invariant. To see this, recall that an infinitesimal dilatation in the boundary gauge theory corresponds to an isometry of the bulk theory with the following transformations:

$$z \rightarrow (1 + \epsilon)z; \quad \rho \rightarrow (1 + \epsilon)\rho; \quad t \rightarrow (1 + \epsilon)t, \quad \theta \rightarrow \theta; \quad \omega \rightarrow \omega(1 - \epsilon) \quad (2.3)$$

which leave the tilde variables  $\tilde{\rho}$  and  $\tilde{z}$  unchanged. This isometry changes the position of the D7 branes  $z_{D7} \rightarrow (1 + \epsilon)z_{D7}$ . By eq. (2.1) this imply a rescaling of the mass parameter  $m_q$  in the dual gauge theory by  $m_q \rightarrow (1 - \epsilon)m_q$ . However quantities like  $J$  which depend only upon the tilde variables  $\tilde{z}_{D7}$  or  $\tilde{z}_0$  but not on  $\omega$  or  $z_{D7} = m_q R^2 / 2\pi\alpha'$  separately, are invariant under this transformations, meaning that they are independent of the mass scale  $m_q$ . More generally, the string equations of motion and their solutions are invariant under this isometry when written in terms of the tilde variables.

Choosing the static gauge  $\tau = t$  and  $\sigma = \tilde{z}$ , the Nambu-Goto action takes the form

$$S = -\frac{R^2\omega}{\pi\alpha'} \int dt \int_{\tilde{z}_{D7}}^{\tilde{z}_0} d\tilde{z} \frac{1}{\tilde{z}^2} \sqrt{(1 - \tilde{\rho}^2)(\tilde{\rho}'^2 + 1)} , \quad (2.4)$$

where a prime denotes a derivative with respect to  $\tilde{z}$  and a factor of 2 has been included to account for the string symmetry around  $\rho = 0$ . The string angular momentum is computed as

$$J = \frac{\partial L}{\partial \omega} = \frac{R^2}{\pi \alpha'} \int_{\tilde{z}_{D7}}^{\tilde{z}_0} d\tilde{z} \frac{1}{\tilde{z}^2} \sqrt{\frac{\tilde{\rho}'^2 + 1}{1 - \tilde{\rho}^2}}. \quad (2.5)$$

The meson energy is the same as the string energy measured by a gauge theory observer, that is

$$E = \omega \frac{\partial L}{\partial \omega} - L = \frac{R^2 \omega}{\pi \alpha'} \int_{\tilde{z}_{D7}}^{\tilde{z}_0} d\tilde{z} \frac{1}{\tilde{z}^2} \sqrt{\frac{\tilde{\rho}'^2 + 1}{1 - \tilde{\rho}^2}}. \quad (2.6)$$

The string profile  $\tilde{\rho}(\tilde{z})$  is determined by solving the following equation of motion:

$$\frac{\tilde{\rho}''}{1 + \tilde{\rho}^2} - \frac{2\tilde{\rho}'}{\tilde{z}} + \frac{\tilde{\rho}}{1 - \tilde{\rho}^2} = 0 \quad (2.7)$$

with the boundary condition stating that the string ends orthogonally on the D7-brane

$$\left. \frac{d\tilde{\rho}}{d\tilde{z}} \right|_{\tilde{z}=\tilde{z}_{D7}} = 0$$

and the following conditions at  $\tilde{z}_0$ , which express the symmetry of the string profile

$$\tilde{\rho}(\tilde{z}_0) = 0, \quad \left. \frac{d\tilde{\rho}}{d\tilde{z}} \right|_{\tilde{z}=\tilde{z}_0} \rightarrow -\infty.$$

The equation of motion and these boundary conditions determine the maximal string penetration  $\tilde{z}_0$  as function of the position  $\tilde{z}_{D7}$  of the D7-brane, and the string profile  $\tilde{\rho}(\tilde{z})$  for  $\tilde{z}_{D7} \leq \tilde{z} \leq \tilde{z}_0$ . We can determine numerically the function  $\tilde{z}_{D7}(\tilde{z}_0)$  in all the interval by numerically solving the differential equation. Nevertheless we can find an *analytical* closed expression for the relation between  $\tilde{z}_{D7}$  and  $\tilde{z}_0$  using interpolation two point Padé approximant. This closed expression is quite simple

$$\tilde{z}_{D7}(\tilde{z}_0) = \frac{\mathcal{C} \tilde{z}_0^3}{1 + \mathcal{C} \tilde{z}_0^2}, \quad (2.8)$$

where  $\mathcal{C} = \frac{\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2}$ . The constant  $\mathcal{C}$  is determined by the asymptotic behaviour of  $\tilde{z}_{D7}$  as presented in [6]. For details see Appendix A.

The spectrum  $E(J)$  can be obtained by solving for the string profile and substituting it back into equations (2.6) and (2.5); this determines  $E(\omega)$  and  $J(\omega)$ , and hence the spectrum in parametric form. Equation (2.7) can be solved numerically for arbitrary  $\omega$ . The solutions that we are interested in are solutions that can be continued past the position of the D7-brane (at which  $\tilde{z} = \omega z_{D7}$  and  $\tilde{\rho}' = 0$ ) to the AdS boundary at  $\tilde{z} = 0$ . Analytic results can be obtained for the spectrum in the two regimes  $\tilde{z}_0 \gg 1$  that correspond to  $J \ll \sqrt{g_s N}$ ,  $\omega \rightarrow \infty$  and  $\tilde{z}_0 \ll 1$  that correspond to  $J \gg \sqrt{g_s N}$  and  $\omega \rightarrow 0$ .

## 2.1 $\tilde{z}_0 \ll 1$ limit

We will show that in this limit the relevant operators have large spin  $J \gg \sqrt{\lambda}$ . The theory has a free parameter  $z_{D7}$  that we can fix as  $z_{D7} = 1$ .

We see from (2.8) (see appendix A) that in this regime

$$\tilde{z}_{D7}(\tilde{z}_0) \sim \mathcal{C} \tilde{z}_0^3 \quad (2.9)$$

so  $\tilde{z}_0 \sim \omega^{1/3}$  and we conclude that the string is rotating with a very slow angular velocity  $\omega \rightarrow 0$ .

The string profile is very near to the static profile  $\tilde{\rho}_{st}$ . The solution to the profile differential equation can be studied via an expansion in powers of  $\tilde{z}$  around the well-known solution  $\tilde{\rho}_{st}(\tilde{z})$  corresponding to a static string [12]. When  $\tilde{z}_0 \ll 1$ , the lower limit  $\tilde{z}_{D7}$  is even smaller.

Since the string attached to the D7-brane is very close to the AdS boundary (in the  $\tilde{z}$  coordinate) and rotates very slowly ( $\omega \rightarrow 0$ ), we expect that a good approximate solution  $\tilde{\rho}(\tilde{z})$  for  $\omega \rightarrow 0$  is a deformation of the static potential solution between two quarks,

$$\tilde{\rho}_{st.}(\tilde{z}) = \int_{\tilde{z}}^{\tilde{z}_0} dx \frac{x^2}{\sqrt{\tilde{z}_0^4 - x^4}}, \quad (2.10)$$

whose first correction  $\tilde{\rho} = \tilde{\rho}_{st.} + \delta\tilde{\rho}$  can be obtained in closed form in terms of elliptic integrals. The energy can be computed to leading order in  $\tilde{z}_0$ . The result reads [6],

$$E \simeq 2m_q \left[ 1 - \frac{\mathcal{C}^2 \tilde{z}_0^2}{2} + \mathcal{O}(\tilde{z}_0^4) \right], \quad \mathcal{C} = \frac{\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2} \quad (2.11)$$

From equation (2.5), the result for the spin  $J$  at leading order in  $\tilde{z}_0$  is

$$J \simeq \frac{R^2 \mathcal{C}}{\pi \alpha \tilde{z}_0} + \mathcal{O}(\tilde{z}_0). \quad (2.12)$$

As  $\tilde{z}_0 \ll 1$  we have  $J/\sqrt{\lambda} \gg 1$  as expected. By eliminating  $\tilde{z}_0$  and restoring  $m_q$ , one gets

$$E = 2m_q - E_b, \quad E_b = m_q \frac{\kappa^4}{4J^2} \quad \kappa^4 = \frac{16\mathcal{C}^4 g_s N}{\pi}. \quad (2.13)$$

Thus, the binding energy  $E_b$  coincides exactly with that of a classical system consisting of two non-relativistic particles of equal masses  $m_q$  bound by a Coulomb potential  $V(\rho) = -\kappa^2/\rho$ . Such agreement can be attributed to the highly non-trivial modification of the open string spectrum on the D7-brane in AdS for large  $J$ . This means that a string in AdS space can not only describe the statics of a Coulomb potential but also the dynamics of masses bound by it. It is interesting to see that in the string calculation, the mass of the quarks, their kinetic energy and the Coulomb energy, all come from the energy of the string.

The asymptotic relations (2.11) and (2.12) are the main results of this section and will be useful in what follows. We will review now the complementary limit  $\tilde{z}_0 \gg 1$ .



## 2.2 $\tilde{z}_0 \gg 1$ limit

To consider the opposite limit  $\tilde{z}_0 \gg 1$ , it is convenient to write the equation of motion in the gauge  $\tilde{\rho} = \sigma$ . Thus the equation of motion of the string profile has the form,

$$\frac{\tilde{z}''}{1 + \tilde{z}'^2} + \frac{2}{\tilde{z}} - \frac{\tilde{\rho}\tilde{z}'}{1 - \tilde{\rho}^2} = 0, \quad (2.14)$$

which is supplemented with the boundary conditions

$$\left. \frac{\partial L}{\partial (X')^M} \delta X^M \right|_{\partial \Sigma} = 0, \quad (2.15)$$

so that the action is stationary. Since  $\delta \tilde{z}|_{\partial \Sigma} = 0$  and  $\delta \tilde{\rho}|_{\partial \Sigma}$  is arbitrary (due to the Neumann boundary condition), we must impose

$$\left. \frac{\partial L}{\partial \tilde{\rho}'} \right|_{\partial \Sigma} = \frac{\tilde{\rho}'}{\tilde{z}^2} \sqrt{\frac{1 - \tilde{\rho}^2}{\tilde{\rho}'^2 + \tilde{z}'^2}} \Big|_{\partial \Sigma} = 0. \quad (2.16)$$

It follows that either  $\tilde{\rho}'|_{\partial \Sigma} = 0$ , which means that the string ends orthogonally on the D7-brane, or  $\tilde{\rho}|_{\partial \Sigma} = 1$ , which means that the endpoints of the string move at the speed of light. The second condition cannot correspond to a bound state of two hypermultiplet quarks because it cannot describe a string with both endpoints on the D7-brane as can be seen by expanding  $\tilde{z}(\tilde{\rho})$  for  $\tilde{\rho} \lesssim 1$  in (2.14). It is possible, however, that a string with these boundary conditions can extend from the D7-brane to the AdS boundary at  $\tilde{z} = 0$ .

In the limit  $\tilde{z}_0 \gg 1$ , we will have large spin  $J \ll \sqrt{\lambda}$  (see below). Fixing  $z_{D7} = 1$  as in the previous subsection, and taking into account (2.8) (see appendix A)

$$\tilde{z}_{D7}(\tilde{z}_0) \sim \tilde{z}_0 \quad (2.17)$$

the string profile remains very near to the position of the D7-brane. As  $\tilde{z}_0 \sim \omega$  we conclude that the string is rotating with a high angular velocity  $\omega \rightarrow \infty$ . One can check by numerical analysis that  $\tilde{z}(\rho) > \omega z_{D7} \simeq \tilde{z}_0$ .

In particular the second term in (2.14) is negligible and the only solution satisfying the appropriate boundary conditions is a constant  $\tilde{z}(\tilde{\rho}) = \omega z_{D7} \simeq \tilde{z}_0$ , which is precisely what one would expect in the flat space limit.

Note that since  $0 < |\tilde{\rho}| < 1$ , after rescaling back we have  $0 < |\rho| < 1/\omega \rightarrow 0$ , namely a very short string insensitive to the AdS curvature. Corrections to this solution come from considering the  $2/\tilde{z}$  term in (2.14) and are of order  $1/\omega z_{D7}$ .

In this case the equations (2.6) and (2.5) give the energy and the spin to leading order in  $1/\omega z_0$  [6]:

$$E \simeq \frac{\pi m_q}{\tilde{z}_0}, \quad J \simeq \frac{\sqrt{\lambda}}{4\tilde{z}_0^2}. \quad (2.18)$$

Since  $\tilde{z}_0 \gg 1$  we have  $J \ll \sqrt{g_s N}$ , as anticipated. Eliminating  $\tilde{z}_0$  we find

$$E \simeq \frac{\sqrt{2}\pi^{3/4}m_q}{(g_s N)^{1/4}}\sqrt{J}. \quad (2.19)$$

Therefore for  $J \ll \sqrt{g_s N}$  the meson masses follow a Regge behaviour.

Summarizing, from the viewpoint of the boundary theory, the meson mass, as a function of  $J$ , follows a Regge trajectory for small  $J$ , whereas for large  $J$  it is well explained by particles moving in a Coulomb potential. Here also the main result of this section, that we will use in what follows, are the asymptotic formulas (2.18).

### 3 Cusp anomalous dimension

In this section we will recall the construction of the cusp anomalous dimension from the definition of the Wilson loop in the context of AdS/CFT. We will follow the notation and conventions given in [13] and [14]. For details about the calculation of the cusp anomalous dimension we refer the reader to the original articles.

In the context of the AdS/CFT, the expectation value of a Wilson loop in the gauge theory is given by the action of a string bounded by the curve at the boundary of space [12],

$$\langle W[C] \rangle = \int_{\partial X = C} \mathcal{D}X \exp(-\sqrt{\lambda} S[X]), \quad (3.1)$$

where  $S[X]$  is the string action. For large  $\lambda$ , we can estimate the path integral by the steepest descent method. Consequently the expectation value of the Wilson loop is related to the area  $A$  of the minimal surface bounded by  $\mathcal{C}$  as

$$\langle W \rangle = \exp\left(-\frac{\sqrt{\lambda}}{2\pi} A(\mathcal{C})\right). \quad (3.2)$$

The construction starts by considering Euclidean AdS space in Poincaré coordinates

$$ds^2 = \frac{1}{z^2}(dx^\mu dx_\mu + dz^2) \quad (3.3)$$

where the conformal boundary is at  $z = 0$ .

The computation of the Wilson loop in *AdS* requires an IR regularization, since the area of the minimal surface terminating at the boundary of *AdS* is infinite due to the factor  $z^{-2}$  in the metric. Thus, in order to make sense of the ansatz (3.2), we need to regularize the area. The standard procedure to regularize the area  $A_\epsilon$  is based on cutting off that part of the surface at which  $z < \epsilon$ . On the gauge theory side, the Wilson loop requires regularization in the ultraviolet. According to the UV/IR relation, the IR cutoff  $\epsilon$  in AdS should be identified with the UV cutoff in the gauge theory. If  $\ell$  is the length of the Wilson loop contour, the area  $A_\epsilon$  is defined by

$$A_\epsilon = \frac{\ell}{\epsilon} + A_{ren} + O(\epsilon).$$

In the presence of a cusp a new logarithmic divergence appears.

$$A_\epsilon = \frac{\ell}{\epsilon} + \Gamma_{cusp}(\theta) \log \epsilon + A_{ren} + O(\epsilon).$$

Consider a loop formed by two half-lines with cusp angle  $\theta$  on the two dimensional plane  $(x_1, x_2)$  defined by the metric (3.3). In this case we can obtain an analytic expression for the minimal surface area whose boundary is the corresponding loop. We use polar coordinates to parametrize the worldsheet,

$$x_1 = \rho \cos \varphi, \quad x_2 = \rho \sin \varphi. \quad (3.4)$$

Using the conformal symmetry  $z \rightarrow \lambda z, x_\mu \rightarrow \lambda x_\mu$ , we can assume that the minimal surface is described by [14]

$$z(\rho, \varphi) = \frac{\rho}{f(\varphi)}, \quad (3.5)$$

with boundary conditions  $z(\rho, 0) = z(\rho, \theta) = 0$ . This symmetry under dilatation is crucial for our proposes because it is a guide to relate the parameters of the Wilson loop with the corresponding parameters in the meson model. The corresponding symmetry for the side of the meson model is (2.3). We need to relate an invariant under dilatation in the two models. A natural candidate is the function  $f(\varphi)$ .

With this factorization of the  $\rho$  and  $\varphi$  dependence, the minimal surface condition, i.e. the equation of motion, is an ordinary differential equation for  $f(\varphi)$  with boundary conditions  $f(0) = f(\theta) = \infty$ . In what follows we will not need the explicit form of this differential equation. The integration can be formally induced if we notice that the translation symmetry ( $\varphi \mapsto \varphi - \alpha$ ,  $\alpha$  an infinitesimal constant) implies the conserved quantity,

$$\mathcal{H} = \mathcal{L} - f' \frac{\partial \mathcal{L}}{\partial f'} = \frac{f^4 + f^2}{\sqrt{f^4 + f^2 + (f')^2}}. \quad (3.6)$$

From the symmetry of the profile  $f$  has a minimum at  $\varphi = \theta/2$  where  $\theta$  is the cusp angle. Using this condition and the above conserved quantity we can write

$$K = f_0 \sqrt{1 + f_0^2}, \quad f_0 = f(\theta/2). \quad (3.7)$$

Hence, the integration yields

$$\theta = 2K \int_{f_0}^{\infty} \frac{df}{\sqrt{(f^4 + f^2)^2 - K^2(f^4 + f^2)}}, \quad (3.8)$$

which fixes the relation between  $f_0$  and the cusp angle  $\theta$ . Indeed,  $\theta(f_0)$  is a monotonically decreasing function of  $f_0$ ,  $\theta(0) = \pi$  and  $\theta(\infty) = 0$  (see Fig. 9 blue line).

The regularized area defined with cutoffs  $z = \rho/f(\varphi) > \epsilon$  and  $\rho < L$  is

$$\begin{aligned} A_{\epsilon,L} &= \int d\rho d\varphi \frac{\sqrt{f^4 + f^2 + (f')^2}}{\rho} \\ &= \frac{2L}{\epsilon} + \Gamma_{\text{cusp}} \log \frac{\epsilon}{L} + A_0(\theta) + \dots \end{aligned}$$

where the dots denote terms vanishing for  $\epsilon \rightarrow 0$  and

$$\Gamma_{\text{cusp}} = 2f_0 - 2 \int_{f_0}^{\infty} \left( \sqrt{\frac{f^4 + f^2}{f^4 + f^2 - K^2}} - 1 \right) df.$$

$A_0$  is a regular term that we will not need in what follows.

The substitution  $f^2 = f_0^2 + \eta^2$  [15] gives

$$\Gamma_{\text{cusp}}(\theta) = \int_{-\infty}^{\infty} d\eta \left( 1 - \sqrt{\frac{1 + \eta^2 + f_0^2}{1 + \eta^2 + 2f_0^2}} \right).$$

The term 1 inside the integral is an infinite subtraction constant which makes the area finite. We can thus identify the logarithmic divergence coefficient with the cusp anomaly. A closed form of this integral can be obtained for  $\Gamma_{\text{cusp}}$ , in terms of a hypergeometric function [13],

$$\Gamma_{\text{cusp}}(\theta) = \frac{\pi}{2} \frac{f_0^2}{\sqrt{1 + f_0^2}} {}_2F_1\left(\frac{1}{2}, \frac{3}{2}, 2, \frac{-f_0^2}{1 + f_0^2}\right). \quad (3.9)$$

This relation between the cusp anomalous dimension and the parameter  $f_0$  and the corresponding relation between  $\theta$  and  $f_0$  given by

$$\theta(f_0) = 2K \int_0^{\infty} \frac{d\eta}{(\eta^2 + f_0^2) \sqrt{(\eta^2 + f_0^2 + 1)(\eta^2 + 2f_0^2 + 1)}}, \quad (3.10)$$

where the change of variable  $f^2 = f_0^2 + \eta^2$  was applied, will be very important in what follows.

## 4 Dual Conformal symmetry of planar $\mathcal{N} = 4$ SYM

A subgroup of the symmetries of the maximally supersymmetric Yang-Mills theory ( $\mathcal{N} = 4$  SYM) with gauge group  $SU(N_c)$  in the planar limit ( $N_c \rightarrow \infty$ ,  $\lambda = g^2 N_c$  held fixed) is a symmetry known as *Dual Conformal Symmetry*. The covariant version of this symmetry can be written explicitly as conformal transformations of the *momentum* space of the theory [1]. This symmetry can be used to compute in a novel

way the spectrum of bound states of massive  $W$  bosons in the theory. By construction this is a gauge theory of massless particles. However, massive particles can be introduced in a natural way via a Higgs mechanism. This allows us to discuss the scattering of massive  $W$  bosons. Their masses can be adjusted by given specific vev's to scalar field expectations values. For details see [1] and refereces there in. *The scattering between these massive particles can be studied by means of the cusp anomalous dimension.* As stated before the cusp anomalous dimension was originally defined in [14] as the logarithmic divergence that arises for a Wilson loop operator when there is a cusp in their contour. A cusp is defined when a straight line makes a sudden turn by an angle  $\theta$ , the cusp angle. The Wilson loop develops a logarithmic divergence, as stated in the previous section. It governs in particular the IR divergences for the scattering of massive particles in the presence of massless gluons. In  $\mathcal{N} = 4$  super Yang Mills, the Regge limit of the four point massive scattering amplitudes on the Coloumb branch is governed by the cusp anomalous dimension. As  $t \gg s, m^2$ , we have that  $\log \mathcal{A} \sim \log t \Gamma_{cusp}(\theta)$  where  $\mathcal{A}$  is the planar amplitude divided by its value a tree level. The reason is the following. Take for example the four point amplitude.

As a consequence of the dual conformal symmetry of the planar SYM theory the four point amplitude factorizes in the product

$$A_4(s, t, m_1, \dots, m_4) = A_4^{tree} \times M(u, v)$$

where  $A_4^{tree}$  is the contribution at tree level and

$$u = \frac{4m_1m_3}{-s + (m_1 - m_3)^2} \quad v = \frac{4m_2m_4}{-t + (m_2 - m_4)^2}$$

There are two ways to make  $v$  small  $v \rightarrow 0$  while keeping  $u$  fixed. These two very different limits give the same result. Take  $m_2 = m_4 = m$  for simplicity, then

$$u = \text{fixed}, \quad v = \frac{4m^2}{-t}$$

On the one hand we can take  $t \rightarrow \infty$ . From Regge theory we know that we need to take the highest spin state at each energy  $E$ . The trajectory is  $j(s)$  and  $s = E^2$ . Here  $j$  is the angular momentum and  $s$  the square of the center of mass energy. <sup>1</sup>

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<sup>1</sup>We can read off the spectrum of a theory from the fall-off of the correlators at large distances. The same fall-off can be read from the amplitude of fast particles at large impact parameter  $b$

$$\lim_{t \rightarrow \infty} A(t, b) \sim \sum_n c_n t^j e^{-m_n b}$$

as  $b \rightarrow \infty$ . In particular, taking only the dominant term we have

$$\lim_{t \rightarrow \infty} A(s, t) \sim t^{j+1}$$

where  $j(s)$  is the leading trajectory.

The analytic continuation of the function  $j(s)$  then determines the behaviour of the amplitude in the ultra relativistic limit  $t \rightarrow \infty$  with  $u$  fixed, through

$$A \sim t^{j(s)+1}$$

(provided only that  $j(s)$  remains the leading trajectory).

We see that in the limit  $t \rightarrow \infty$ , the amplitude, with all other variables held fixed, is equivalent to the limit  $m \rightarrow 0$ . In this limit the amplitude is known to become infrared divergent and its leading terms are governed by the anomalous dimension  $\Gamma_{cusp}$  of a Wilson loop with a cusp,  $A \sim m^{\Gamma_{cusp}}$ .

Identifying the exponents in the two asymptotic limits we thus find<sup>2</sup>

$$j(s) + 1 = - \left( -\frac{\sqrt{\lambda}}{2\pi} \right) \Gamma_{cusp}(\theta), \quad s = 4m^2 \cos^2 \left( \frac{\theta}{2} \right) \quad (4.1)$$

where  $s = E^2$  is the energy in the CM frame,  $j(s)$  is a Regge trajectory and  $\Gamma_{cusp}$  is the cusp anomalous dimension associated with a Wilson loop with cusp angle  $\theta \in [0, \pi]$ . This relation has been derived and used previously in Ref. [4], to which we refer the reader for more details. A similar relation is known to give the infrared-divergent part of the gluon trajectory as  $m^2 \rightarrow 0$  [16], but we stress that in planar  $\mathcal{N} = 4$  SYM, Eq. (4.1) holds for the complete function of  $s/m^2$ . These relations were checked up to three loops in [4].

In the following section we will give analytical support to the validity of this novel duality (4.1) in the context of the AdS/CFT correspondence. First, in section 5, we recover the asymptotic information of the meson model in two asymptotic analytical limits and in section 6 we will give complete support for the validity of this duality for the complete interval, not just the asymptotic limits considered in sections 2.1 and 2.2.

## 5 Asymptotic matching between meson model and cusp anomalous dimension

In this section we will present the main result of our paper. Our first question is if we can match the asymptotic results that comes from the meson model, eqs. (2.11), (2.12) and (2.18) with the corresponding asymptotic results for the angular momentum and energy that comes from the duality (4.1). The numerical evidence presented in [1] shows that, in fact, the matching is possible, at least numerically. This numerical matching is very surprising because the two approaches are quite

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<sup>2</sup> Here we write the duality in the AdS/CFT context, that is the reason why the factor  $\sqrt{\lambda}$  appears in front of  $\Gamma_{cusp}$ . The duality can also be used in the perturbative regime and the dependence on  $\lambda$  is quite different.

different. The aim of this section is to show explicitly that the match between the asymptotic results can be obtained in *analytical form*. The first observation is that the cusp angle and the cusp anomalous dimension depends parametrically on one parameter  $f_0$  of the minimal worldsheet dual to the Wilson loop. From the other hand the meson angular momentum and energy depends on a parameter of the worldsheet of the string profile dual to the meson, namely  $\tilde{z}_0$ . Notice that these two parameters are invariant under dilatations that is an isometry of the respective metrics. The question can be reformulated as if we can construct a relation between these two different parameters. The answer is in the affirmative and in fact the relation is quite simple.

$$f_0 \rightarrow \frac{1}{\tilde{z}_0} \quad (5.1)$$

The central result of our paper is that we can calculate the energy and the angular momentum of the meson model in *closed form* in terms of only one parameter  $\tilde{z}_0$  that is the maximum penetration of the string profile in AdS of the meson model. To obtain the result, that we announced previously in the Introduction, we make use of the parameter  $\tilde{z}_0$  in the equation (3.9) and (3.10) in place of  $f_0$  and apply the duality (4.1). We quote again the result,

$$E(\tilde{z}_0)/m_q = 2 \cos(\theta(\tilde{z}_0)/2) \quad (5.2)$$

$$J(\tilde{z}_0) + 1 = \frac{\sqrt{\lambda}}{4\tilde{z}_0\sqrt{1+\tilde{z}_0^2}} {}_2F_1\left(\frac{1}{2}, \frac{3}{2}, 2, -\frac{1}{\sqrt{1+\tilde{z}_0^2}}\right) \quad (5.3)$$

where

$$\theta(\tilde{z}_0) = 2K \int_0^\infty \frac{d\xi}{(1+\xi^2)\sqrt{(1+\tilde{z}_0^2+\xi^2)(2+\tilde{z}_0^2+\xi^2)}} \quad (5.4)$$

and  $K = \tilde{z}_0\sqrt{1+\tilde{z}_0^2}$ .  ${}_2F_1$  is the hypergeometric function. By using these formulas the reconstruction of the meson spectra is straightforward. In particular, we can write the energy and angular momentum in parametric form with parameter  $\tilde{z}_0$  to *any order* in powers of  $\tilde{z}_0$ . A parametric plot of these basic observables is the  $E - J$  graph reported in [6] and reproduced elsewhere. Our graph (see Fig. 3) obtained with the aid of equations (5.2) and (5.3) coincides with previously reported plot using numerical methods [1] so we provide analytical support for the proposed conformal duality (4.1) in the context of AdS/CFT.

As a check we can recover the asymptotic analytical information given in [6] and reproduce the asymptotic limits. Taking  $\tilde{z}_0 \gg 1$ , (equivalent to take  $f_0 \ll 1$ ) we get

$$\theta(\tilde{z}_0) = -\pi \frac{1}{\tilde{z}_0} + \pi \cdots \quad (5.5)$$

On the other hand, the corresponding asymptotic limit for the cusp anomalous dimension for the parameter  $\tilde{z}_0 \gg 1$  is

$$J(\tilde{z}_0) + 1 = \frac{\sqrt{\lambda}}{2\pi} \left( \frac{\pi}{2} \frac{1}{\tilde{z}_0^2} - \frac{7}{16} \pi \frac{1}{\tilde{z}_0^2} + \dots \right). \quad (5.6)$$

The  $\tilde{z}_0 \gg 1$  limit corresponds to  $f_0 \rightarrow 0$  and then to a cusp angle  $\theta$  near  $\pi$  (no scattering). To leading order in  $1/\tilde{z}_0$  we have

$$\frac{E}{m} = 2 \cos \left( \frac{\theta}{2} \right) \approx \pi - \theta \quad (5.7)$$

and

$$\frac{E}{m} \simeq \pi \frac{1}{\tilde{z}_0}, \quad J + 1 \simeq \frac{\sqrt{\lambda}}{4} \frac{1}{\tilde{z}_0^2}, \quad (5.8)$$

by using our results (5.2, 5.3) and the expansion (B.1) just to leading order in the parameter  $\tilde{z}_0$ . Perfect agreement with the previously reported formulas is evident. We could predict the next-to-next to leading order behaviour of the meson as is clear from our closed relations.

The other interesting limit is  $\tilde{z}_0 \ll 1$ . In this limit  $f_0 \rightarrow \infty$  and the corresponding cusp angle  $\theta \rightarrow 0$ . From the expression (B.2) we have to leading order in  $\tilde{z}_0$  (see appendix B),

$$\theta(\tilde{z}_0) = \frac{2\pi^{1/2}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \tilde{z}_0, \quad (5.9)$$

For the energy, as the cusp angle is small, we can use a simple expansion

$$\frac{E}{m} = 2 \cos \left( \frac{\theta}{2} \right) \approx 2 - \frac{\theta^2}{4} = 2 - \pi \left( \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right)^2 \tilde{z}_0^2 = 2 - \pi \left( \frac{2\pi^2}{\Gamma(\frac{1}{4})^4} \right) \tilde{z}_0^2 \quad (5.10)$$

where we have used the identity  $\Gamma(\frac{3}{4})\Gamma(\frac{1}{4}) = \sqrt{2}\pi$ . At first sight this is not the result obtained from the analysis of the corresponding asymptotic limit from the meson (2.11), unless

$$\mathcal{C}^2 = \pi \left( \frac{2\pi^2}{\Gamma(\frac{1}{4})^4} \right).$$

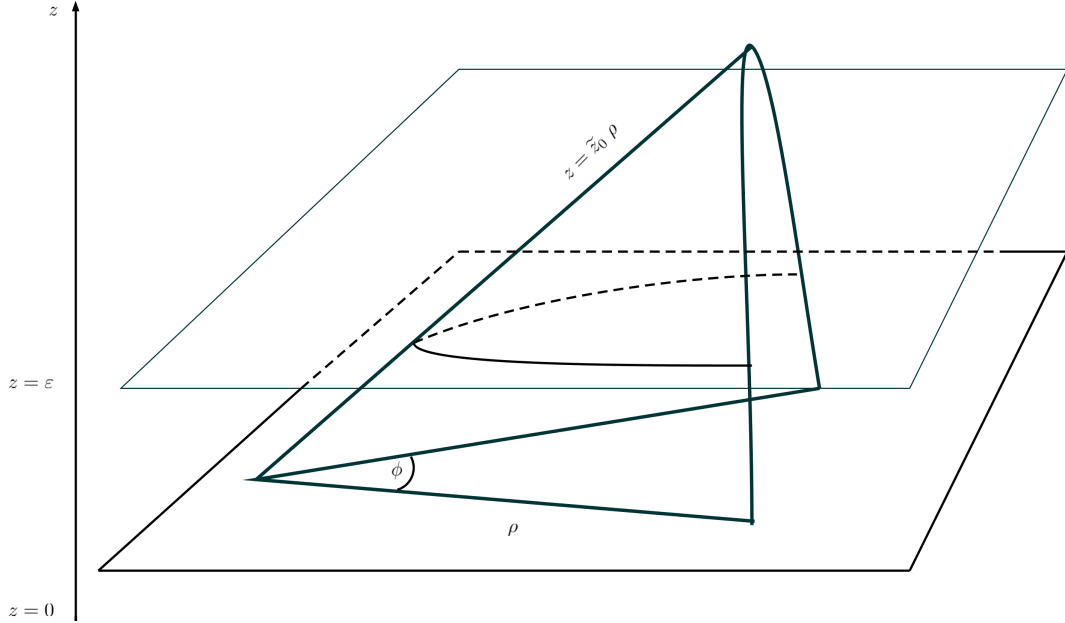
Using the definition  $\mathcal{C} = \frac{\sqrt{2}\pi^{3/2}}{\Gamma(1/4)^2}$ , we can verify that this relation is in fact valid. So we conclude that our proposal recovers the correct analytical information that we can extract from the meson model. We stress that our relation gives much more than the asymptotic behaviour of the meson model. For the angular momentum we obtain

$$J(\tilde{z}_0) + 1 \simeq \sqrt{\frac{\lambda}{\pi}} \frac{\Gamma(\frac{3}{4})}{4\Gamma(\frac{5}{4})} \frac{1}{\tilde{z}_0} = \frac{\sqrt{2\pi\lambda}}{\Gamma(\frac{1}{4})^2} \frac{1}{\tilde{z}_0}, \quad (5.11)$$



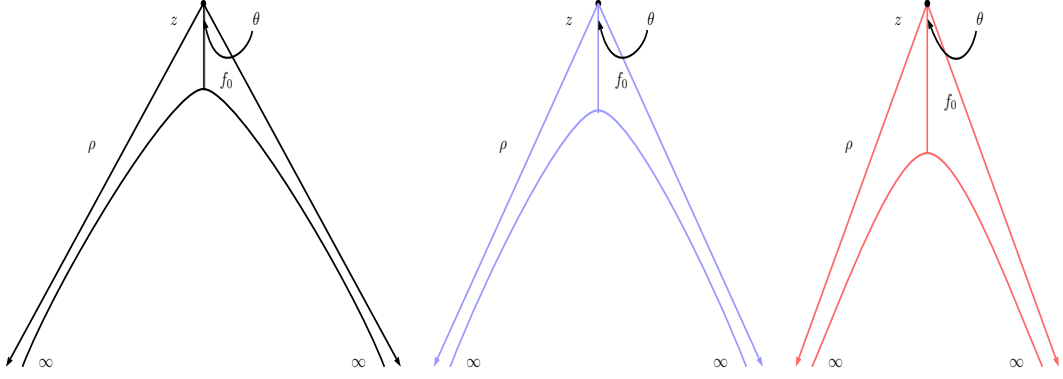
where we have used the identity  $\frac{\Gamma(\frac{3}{4})\Gamma(\frac{1}{4})^2}{\Gamma(\frac{5}{4})} = 4\sqrt{2}\pi$ . This result coincides with the meson model using the definition of  $\mathcal{C}$ .

To grasp the meaning behind the identification (5.1) recall that for a given cusp angle  $\theta$ ,  $f_0$  is the minimum of the function  $f$  defined in (3.5), and by symmetry considerations it is  $f_0 = f(\theta/2)$ . Consider a plane with fixed  $z$ , then  $f_0$  is proportional to the distance from the cusp to the minimum of the curve  $\rho \sim f(\varphi)$  (see Fig. 1 and Fig. 2).  $\theta$  is a monotonically decreasing function of  $f_0$  starting from  $\theta = \pi$ ,  $f_0 = 0$  and going to  $\theta \rightarrow 0$  as  $f_0 \rightarrow \infty$  (see Fig. 9 blue curve). Then according to our dictionary  $\tilde{z}_0 \rightarrow \infty$  as  $\theta \rightarrow \pi$  and  $\tilde{z}_0 \rightarrow 0$  as  $\theta \rightarrow 0$ . In the first case the behaviour of the meson is governed by a Regge trajectory  $E \sim \sqrt{J}$  and in the second case the energy is of the order  $E \sim 2m_q$ , the binding energy. So the cusp angle has a very nice interpretation in terms of our dictionary. When the Wilson loop is nearly smooth the behaviour of the meson model is the Regge limit and when the cusp is very spiky the meson has a nearly vanishing binding energy.



**Fig. 1:** The minimal surface based on the Wilson loop with a cusp. We also show the regularization parameter  $\epsilon$ .

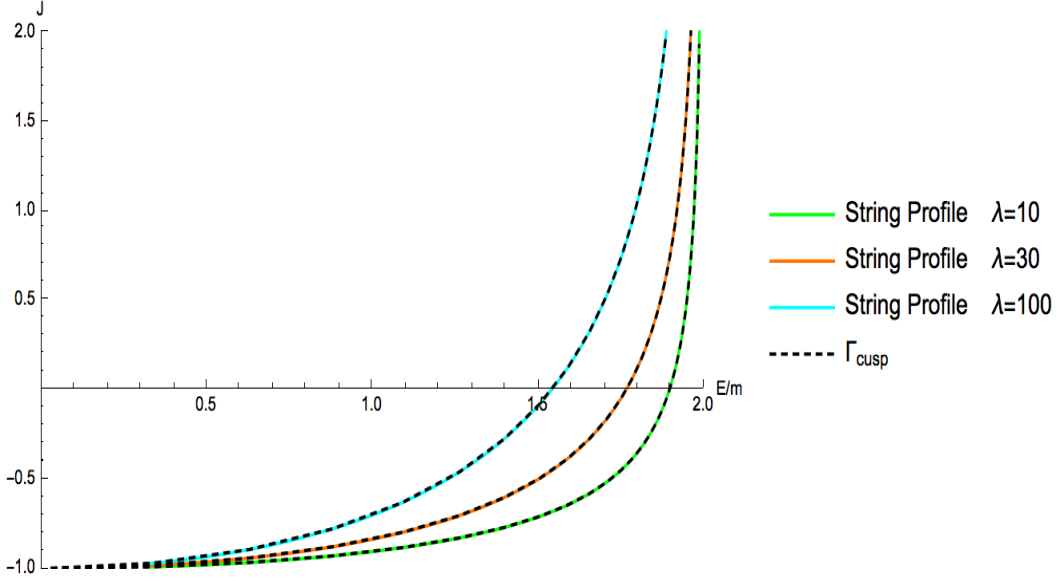
It is also interesting to notice that in the limit  $\tilde{z}_0 \rightarrow \infty$  as  $\theta \rightarrow \pi$  the Wilson loop is very smooth and the Bremsstrahlung function  $B(N, \lambda)$  [17] can be used as an exact calculation for any  $N_c$  and  $\lambda$  for the cusp anomalous dimension  $\Gamma_{cusp}$ . The



**Fig. 2:** A transverse cut of the minimal surface at  $z = \text{const.}$ . The geometric meaning of the parameter  $f_0$  is proportional to the distance of the cusp vertex to the minimal value of the function  $f(\varphi)$ . We show diverse profiles of the function  $f$  as we vary the cusp angle  $\theta$ .

relation is

$$\Gamma_{\text{cusp}}(\theta - \pi) = B(\theta - \pi)^2 + O(\theta^4).$$



**Fig. 3:** Regge trajectories of  $\mathcal{N} = 4$  SYM for  $\lambda = 10, 30, 100$ . The color lines are numerical results from the meson model and the black curve is the analytical result (5.2, 5.3).

We obtained these formulas using the results of the Wilson loop with a cusp angle  $\theta$  and the conformal duality reported in section 4 by a simple application of the dictionary (5.1). The results given are valid in the context of AdS/CFT duality with planar  $\mathcal{N} = 4$  SYM theory.

## 6 Exploring the relation between $f_0$ and $\tilde{z}_0$ beyond the asymptotic limits

In this section we will show that the relation between the two very different approaches, the Wilson loop with a cusp and the meson model given by the identification of the maximum string penetration  $\tilde{z}_0$  and the minimum value of the function  $f(\varphi)$ ,  $\varphi \in (0, \theta)$ ,  $f_0 = f(\frac{\theta}{2})$  with  $\theta$  the cusp angle given by (5.1) is also valid for the entire interval between the asymptotic limits considered in the previous sections.

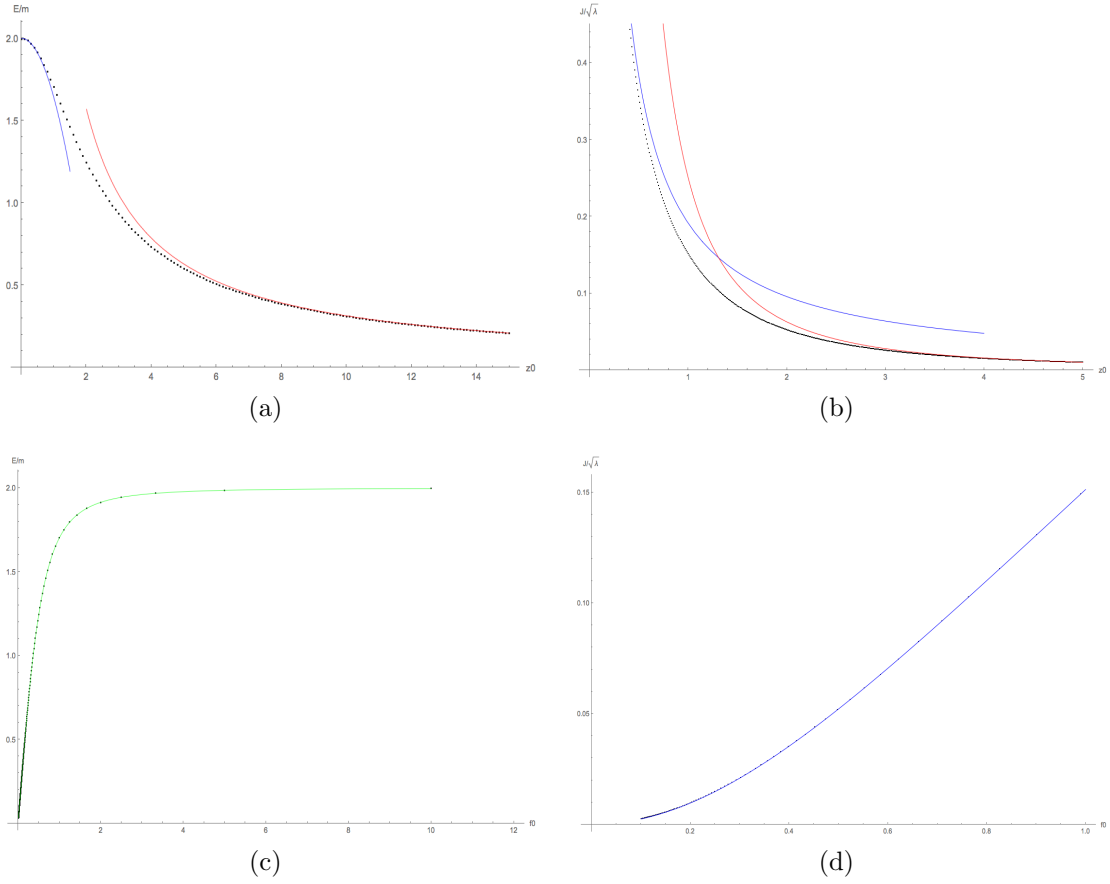
From the fact that the asymptotic limits (5.8, 5.10, 5.11) can be described with the same reparametrization we could argue that the reparametrization is universal, i.e. is valid for the entire interval. But a more compelling approach is to show it with the aid of the plots in Fig. 4. In the plots (a) and (b) we show the numerical integration of the NG equation of motion for the string profile and then we integrated numerically the equations (2.5) and (2.6) to obtain  $E(\tilde{z}_0)$ ,  $J(\tilde{z}_0)$  for the meson model. The Fig. 4 (a) and (b) also show the corresponding asymptotic limits as reported in the original work [6] and that we reproduce here in eqs. (2.11, 2.12) and (2.18). The dotted lines in these graphs are the numerically evaluated corresponding observable in the complete interval.

Next, in Fig. 4 (c) and (d), we plot the corresponding graphs for the same observables  $E/m_q$  and  $J/\sqrt{\lambda}$  but now using the results from the Wilson loop with a cusp, the cusp anomalous dimension  $\Gamma(f_0)$ , and  $\theta(f_0)$  and the duality (4.1). In this way we obtain the energy and angular momentum as functions of  $f_0$ , the minimum of the profile function  $f(\varphi)$ . *Now, the dots come from our previous graphs, evaluated at  $1/f_0$  as predicted by our dictionary.* The full lines in colour comes from the evaluation of the integral for  $\theta(f_0)$  and the hypergeometric function for  $\Gamma_{cusp}$  given in (3.8) and (3.9) respectively, and then using the duality relation to obtain  $E(f_0)$ ,  $J(f_0)$ .

The agreement is eloquent! So we conclude that our dictionary gives a real complete description of the meson model using information from the cusp anomalous dimension and the angle of the corresponding cusp  $\theta$ <sup>3</sup>. The case that we have at hand was scrutinized by us from using interpolation and give excellent results. Nevertheless these interpolation methods can not be used to prove conclusively that the interpolated observables are in fact unique (we have a landscape of possible Padé approximants). And/or that the behaviour of the function must be exactly controlled by the interpolation.

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<sup>3</sup>This result could come as a surprise at least by the following reasons. This is an example that in some cases the asymptotic behaviour determines the complete function for an observable in the gauge theory. In the context of AdS/CFT our experience shows that using two point Padé approximants to interpolate between the asymptotic behaviour of an observable give a surprising good agreement with the exact result (when the exact result is known) and/or with the general consideration of the behaviour of the corresponding function between the asymptotic limits under consideration [18, 19].



**Fig. 4:** (a) The energy as function of the parameter  $\tilde{z}_0$  for the meson model. The asymptotic limits are plotted in blue for  $\tilde{z}_0 \ll 1$  and red for  $\tilde{z}_0 \gg 1$ . The dotted curve corresponds to the numerical integration of the profile of the string and numerical evaluation of the integral (2.6). (b) The angular momentum as function of the parameter  $\tilde{z}_0$  for the meson model. The asymptotic limits are plotted in blue for  $\tilde{z}_0 \ll 1$  and red for  $\tilde{z}_0 \gg 1$ . The dotted curve correspond to the numerical integration of the profile of the string and numerical evaluation of the integral (2.5). (c) In green, the energy as function of the parameter  $f_0$ ,  $E/m_q = 2 \cos(\theta(f_0))$ . The dotted curve correspond to the numerical integration of the profile of the string and numerical evaluation of the integral (2.6) as function of  $f_0$  after using our dictionary (5.1). (d) The angular momenta as function of  $f_0$ . The blue curve is the the plot of the relation (5.3). The dotted curve is an numerical integration of the profile of the string and the numerical evaluation of the integral (2.5) in terms of  $\tilde{z}_0$ . Using our dictionary (5.1) the agreement is evident.

The other reason comes from the fact that two very different constructions, the rotating string on the one hand and the Wilson loop with a cusp on the other, are related using a very simple dictionary. From the side of the Wilson loop it is not clear where the complete profile of the string is encoded. From the side of the meson model it is not clear how the knowledge of the cusp of the Wilson loop and the cusp

anomalous dimension determines the basic observables of the gauge theory to any order in the relevant parameters. That means that we are solving the differential equation for the string profile extending the asymptotic analysis that is reported in the current literature.

As we noted, the full lines (in colour) in Fig. 4 (c) and (d), are the plots of the numerical evaluation of the duality relation using information provided by the Wilson loop approach. So we do not need to know the complete profile of the string in the meson model to obtain the observables  $E$  and  $J$ . What we need is only one point of the complete profile namely  $\tilde{z}_0$  the maximum penetration of the string in the bulk that is also related to the position of the D7 brane  $\tilde{z}_{D7}$  through (A.1).

We also observe that the plot of the energy vs. angular momentum ( $E - J$  graph reported previously in [1] using numerical methods, see Fig. 5) does not change by the simple argument that the dictionary (5.1) is just a change in the parametrization of  $E$  and  $J$ . We can use as a parameter  $f_0$  and  $E(f_0), J(f_0)$  to obtain a point in this plot or we can use  $\tilde{z}_0$  as a parameter,  $E(\tilde{z}_0), J(\tilde{z}_0)$  to obtain the complete plot and of course the graphs will be exactly the same. So we can reproduce the previous numerical analysis with our simple dictionary. As a consequence of this fact we can see that from the side of the meson model we need to integrate a complete profile of the string that is a solution of the NG equation of motion to obtain one point in this graph. Now we need only to know the maximum penetration of the profile (and not the complete profile!) to construct the entire plot  $E - J$ .

## 7 Final comments

We have proposed a new recipe for computing  $E$  and  $J$  for the meson spectrum using information from  $\Gamma_{cusp}(\theta)$  and the duality (4.1) in the context of strong coupling, for the  $\mathcal{N} = 4$  or  $\mathcal{N} = 2$  SYM theories, which have large spin, and whose gravity duals are semiclassical strings rotating in the  $\text{AdS}_5 \times \text{S}_5$  space-time (possibly supplemented by D7-branes). Specifically, we constructed an explicit dictionary that relate the meson model observables with the observables of the Wilson loop with a cusp. We have a prediction at *any* order in  $\tilde{z}_0$  for the observables of the meson model. Implicitly we have solved a differential equation (the profile of the string in the meson model) in terms of closed functions  $\theta(f_0)$  and  $\Gamma_{cusp}(\theta)$  that depend only on the point  $\tilde{z}_0$ .

We have covered the case when the Wilson loop is in the fundamental representation, a “quark” with mass  $m_q$ . Could be also worth explore the generalisation to the case of D3-branes [20, 21] (symmetric representation) and D5-brane (antisymmetric representation). We plan to investigate this issue in a future work.

A holographic computation of the entanglement entropy in conformal field theories has been proposed via the AdS/CFT correspondence [22]. In [23] the authors examine the strong subadditivity constraint via direct calculations. As an example

they work out the Wilson loop with a cusp and confirm strong subadditivity. The calculation is analogous to the one reviewed here (see Section 3). It is interesting to note that this implies also via the dictionary (5.1) that the quark-antiquark pair is entangled. In fact, in [23] the authors find that the entanglement entropy can be obtained as a direct application of the procedure to find  $\Gamma_{cusp}$ . As we know, the total area is

$$A/R^2 = 2L/\epsilon - \Gamma_{cusp}(\theta) \log L/\epsilon + (\text{finite terms}).$$

where  $R$  is the AdS radius.

In [22], it is claimed that the entanglement entropy can be computed as follows

$$S_A = \frac{A(\Sigma)}{4G_N^{(d+2)}}$$

where  $A(\Sigma)$  denotes the area of the surface  $\Sigma$ , and  $G_N^{(d+2)}$  is the Newton constant in  $d+2$  dimensional AdS space. The  $d$  dimensional surface  $\Sigma$  is determined in such way that is the minimal area surface whose boundary coincides with the boundary of the submanifold  $A$ . Thus the entanglement entropy can be computed as (up to constant terms)

$$S_A = \frac{R^2}{4G_N^4} \left( 2L/\epsilon - \Gamma_{cusp} \log L/\epsilon \right)$$

So our guess is that entanglement entropy of the quark anti-quark pair can be written in terms of the spin  $J(\tilde{z}_0)$  of the meson in the gauge theory. In this way the entanglement entropy depends only on the parameter  $\tilde{z}_0$  of the dual string.

Recently in [24] the authors found that the entanglement entropy of three-dimensional conformal field theories contains a universal contribution coming from corners in the entangling surface. This calculation is also very similar to that of  $\Gamma_{cusp}$ . It could be interesting to study these results from the point of view our dictionary to obtain (universal?) physical information encoded in the meson model.

A fundamental understanding of the proposed duality is still lacking. We have no rigorous justification for the interpretation or a convincing geometrical argument for how it works out so well. To answer such question we could need to relate the two worldsheets and not only the two parameters  $\tilde{z}_0$  and  $f_0$ . Also, we do not know whether it would make sense, for more general strings, undergoing some complicated motion (e.g. an infinite string with one endpoint attached to an accelerated heavy quark), and what should be the physical interpretation in that case.

We hope that such questions will trigger further investigations, leading to conceptual clarifications and to new results.

For completeness, let us finally mention that corresponding to open string configurations with large angular momentum have been considered within the AdS/CFT correspondence, but for a different space-time geometry, and/or different dynamical situations. It is natural to wonder if these cases can be approached from the same point of view of this article.

## 8 Acknowledgements

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## Appendix A Relation between $\tilde{z}_{D7}$ and $\tilde{z}_0$ beyond the asymptotic limits

In order to obtain information about the relation of the parameter  $z_{D7}$  and  $z_0$  of the meson model we will show here that in fact this two parameter are related. For that end we will use an interpolating two point Padé approximant that fit the numerical data from the numerical integration of the equation for the string profile  $\tilde{\rho}(\tilde{z})$ , (2.7). The position of the D7-brane can be obtained using the boundary condition  $\tilde{\rho}'(\tilde{z}_{D7}) = 0$  and the point  $\tilde{z}_0$  of maximal penetration of the profile into the bulk from  $\tilde{\rho}(\tilde{z}_0) = 0$ . With these two conditions we get the pair  $(\tilde{z}_{D7}, \tilde{z}_0)$  that we plot as a dotted curve in Fig. 5. Using the interpolating Padé approximant we obtain a nice estimate of the relation between these two parameters given by

$$\tilde{z}_{D7}(\tilde{z}_0) = \frac{\mathcal{C}\tilde{z}_0^3}{1 + \mathcal{C}\tilde{z}_0^2}. \quad (\text{A.1})$$

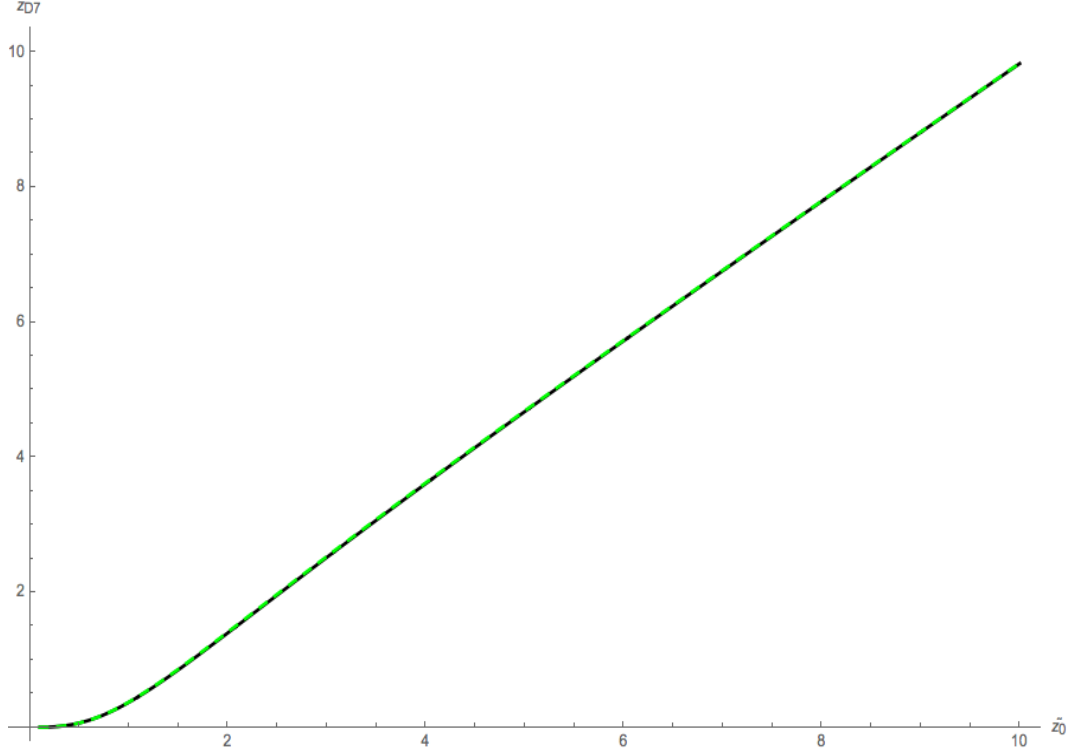
where  $\mathcal{C}$  is defined in the main text. The plot of this relation is represented in Fig. 5, yellow curve. The agreement is excellent! We can now compute two different limits of this relation that are crucial for the meson model asymptotic limits. For  $\tilde{z}_0 \gg 1$  we have  $\tilde{z}_{D7} \sim \tilde{z}_0$  and for  $\tilde{z}_0 \ll 1$  we have  $\tilde{z}_{D7} \sim \mathcal{C}\tilde{z}_0^3$  (see eqs. (2.17) and (2.9) respectively in the main text). Thus, all the relevant information about  $\tilde{z}_{D7}(\tilde{z}_0)$  is completely captured by the asymptotic expansions.

## Appendix B Asymptotic limits for the cusp angle integral (3.10)

We can estimate the value of  $\theta(f_0)$  of the integral (3.10) in two relevant limits. These asymptotic behaviour is important because we can make contact with the corresponding limits of the meson energy and angular momenta through our dictionary (5.1). The limit  $f_0 \rightarrow 0$  of the integral

$$\theta(f_0) = 2K \int_0^\infty \frac{d\eta}{(\eta^2 + f_0^2)\sqrt{(\eta^2 + f_0^2 + 1)(\eta^2 + 2f_0^2 + 1)}},$$

can be estimated using an appropriate Taylor expansion observing that any divergence is an artefact of the approximation. As we already has been observed the



**Fig. 5:** Approximant  $\mathcal{P}_2^3(\tilde{z}_0)$  (green) and the numerical results for  $\tilde{z}_{D7}(\tilde{z}_0)$ .

integral is a smooth function of the parameter  $f_0$ , in fact is a monotonically decreasing function of  $f_0$ . In the limit  $f_0 \ll 1$ , after exploring the general behaviour of the integrand around  $f_0 \rightarrow 0$  we can estimate

$$\theta(f_0) = -\frac{36305\pi f_0^9}{16384} + \frac{359\pi f_0^7}{256} - \frac{61\pi f_0^5}{64} + \frac{3\pi f_0^3}{4} - \pi f_0 + \pi \dots \quad (\text{B.1})$$

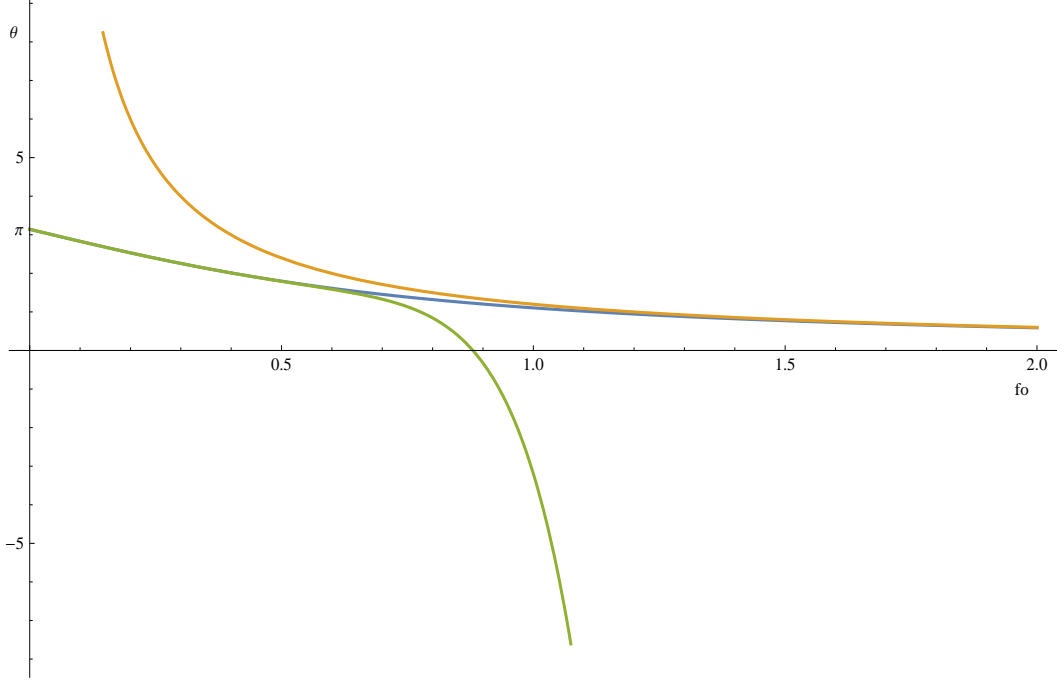
To show explicitly that this estimation works very well see Fig. 6 (green curve).

From the other hand in the limit  $f_0 \gg 1$  the integral (3.10) behaves like

$$\theta(f_0) = \frac{2\pi^{1/2}\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \frac{1}{f_0} - \frac{0.08}{f_0^3} + \frac{0.005}{f_0^5} + \mathcal{O}(\frac{1}{f_0^7}), \quad (\text{B.2})$$

The estimates (B.1, B.2) covers practically all the intermediate values of the integral as can be appreciated in Fig. 6.





**Fig. 6:** The numerical result of the integral (3.10) is in blue. The other graphs are the estimation of the asymptotic behaviour of the integral given in (B.1,B.2).

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